Probability & Statistics (1)

Properties of Expectation (I)

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Introduction

- 在這個章節，我們會介紹更多期望值的屬性；在之前我們提過，如果隨機變數X為discrete random variable的話，其期望值為:

$$E[X] = \sum_{x} xp(x)$$

- 如果X為continuous random variable的話，其期望值為:

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

- 因為$E[X]$為加權平均後的所有可能的隨機變數X，故X一定會介於a與b之間，其期望值也會符合這個性質，

$$P\{a \leq X \leq b\} = 1, a \leq E[X] \leq b$$

Introduction

- 為了確認這個性質，假設X為一個discrete random variable且符合$P\{a \leq X \leq b\} = 1$。因為對於所有x超出$[a, b]$範圍以外的機率都是$p(x) = 0$，故

$$E[X] = \sum_{x:p(x)>0} xp(x) \geq \sum_{x:p(x)>0} ap(x) = a \sum_{x:p(x)>0} p(x) = a$$

[加分題]

You can prove $E[X] \leq b$ in the same manner.

Expectation of Sums of Random Variables

- **Proposition 1**

If X and Y have a joint probability mass function $p(x, y)$, then

$$E[g(X,Y)] = \sum_{y}\sum_{x} g(x,y)p(x,y)$$

If X and Y have a joint probability density function $f(x, y)$, then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

Expectation of Sums of Random Variables

Proof:

When the random variables X and Y are jointly continuous with joint density function $f(x,y)$ and when $g(X,Y)$ is a nonnegative random variable. Because $g(X,Y) \geq 0$, we have

$$E[g(X,Y)] = \int_0^\infty P\{g(X,Y) > t\}dt$$

$$P\{g(X,Y) > t\} = \iint_{(x,y):g(x,y)>t} f(x,y)dydx$$

$$E[g(X,Y)] = \int_0^\infty \iint_{(x,y):g(x,y)>t} f(x,y)dydxdt$$

Expectation of Sums of Random Variables

$$E[g(X,Y)] = \int_0^\infty \iint_{(x,y):g(x,y)>t} f(x,y)dydxdt$$

Interchanging the order of integration gives

$$E[g(X,Y)] = \int_x \int_y \int_{t=0}^{g(x,y)} f(x,y)dtdydx$$

$$= \int_x \int_y g(x,y)f(x,y)dydx$$

Expectation of Sums of Random Variables

- **範例一**

假設某一個交通事故發生在長L的道路上的點X，再發生意外的時候，救護車在點Y，且X與Y都是uniformly distributed在這條道路上。我們假設X與Y相互獨立，試問意外發生地與救護車的期望距離為何?

Solution:

本題要求的就是$E[|X-Y|]$，X與Y的joint density function為

$$f(x,y)=\frac{1}{L^2}, where\ 0<x<L\ and\ 0<y<L$$

Expectation of Sums of Random Variables

從Proposition 1可以得知

$$E[|X-Y|]=\frac{1}{L^2}\int_0^L\int_0^L|x-y|dydx$$

$$\int_0^L|x-y|dy=\int_0^x(x-y)dy+\int_x^L(y-x)dy$$

$$=\frac{x^2}{x}+\frac{L^2}{2}-\frac{x^2}{2}-x(L-x)=\frac{L^2}{2}+x^2-xL$$

$therefore,$

$$E[|X-Y|]=\frac{1}{L^2}\int_0^L\left(\frac{L^2}{2}+x^2-xL\right)dx=\frac{L}{3}$$

Expectation of Sums of Random Variables

- 從Proposition 1可以得到一個有趣的性質
- Suppose that $E[X]$ and $E[Y]$ are both finite and let $g(X,Y) = X + Y$. Then, in the continuous case,

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{\infty}^{\infty}xf(x,y)dydx + \int_{-\infty}^{\infty}\int_{\infty}^{\infty}yf(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}xf_X(x)dx + \int_{-\infty}^{\infty}yf_Y(y)dy = E[X] + E[Y]$$

$$whenever\ E[X]\ and\ E[Y] are\ finite. E[X+Y] = E[X] + E[Y]$$

Expectation of Sums of Random Variables

Suppose that, for random variables X and Y,

$$X \geq Y$$

That is, for any outcome of the probability experiment, the value of the random variable X is greater than or equal to the value of the random variable Y. Since $x \geq y$ is equivalent to the inequality $X - Y \geq 0$, it follows that $E[X - Y] \geq 0$, or, equivalently,

$$E[X] \geq E[Y]$$

Using the result of the previous slide, we may show by a simple induction proof that if $E[X_i]$ is finite for all $i = 1, 2, \ldots, n$, then

$$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$$

Expectation of Sums of Random Variables

- **範例二**

Sample Mean

令X_1, …, X_n為independent and identical distributed random variables，其分布函數為F與期望值μ。使得一個隨機變量序列可被稱為構成分佈 F 的樣本。

$$\bar{X} = \sum_{i=1}^{n} \frac{1}{n} X_i$$

這個被稱為樣本平均數(sample mean)。試問: $E[\bar{X}]$

Solution:

$$E[\bar{X}] = E\left[\sum_{i=1}^{n} \frac{1}{n} X_i\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \mu, since\ E[X_i] \equiv \mu$$

Expectation of Sums of Random Variables

- **Sample Mean**
- The expected value of the sample mean is μ, the mean of the distribution. When the distribution mean μ is unknown, the sample mean is often used in statistics to estimate it.

Expectation of Sums of Random Variables

Boole's inequality

- **範例三**

令$A_1, \ldots, A_n$為事件，其對應的隨機變數為$X_i, i = 1,2, \ldots, n$

$$X_i = \begin{cases} 1 & if\ A_i\ occurs \\ 0 & otherwise \end{cases}$$

使得

$$X = \sum_{i=1}^{n} X_i$$

所以X可以被定義為有多少次的事件(A_i)會發生

$$Y = \begin{cases} 1 & if\ X \geq 1 \\ 0 & otherwise \end{cases}$$

Expectation of Sums of Random Variables

如果最少一次A_i事件發生的話， Y就會等於1

$X \geq Y \Rightarrow E[X] \geq E[Y]$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P(A_i)$$

$$E[Y] = P\{at\ least\ one\ of\ the\ A_i\ occur\} = P\left(\bigcup_{i=1}^{n} A_i\right)$$

此時我們可以得出Boole’s inequality

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Expectation of Sums of Random Variables

- **範例四**

Moments of binomial random variables

令X為binomial random variable，參數為n與p。代表著當成功的機率為p的時候，在n次的獨立試驗中成功了幾次。試問: $E[X]$

Solution:

$$X = X_1 + X_2 + \cdots + X_n, where\ X_i = \begin{cases} 1 & if\ the\ ith\ trial\ is\ a\ success \\ 0 & if\ the\ ith\ trial\ is\ a\ failure \end{cases}$$

Hence, X_i is a Bernoulli random variable having expectation $E[X_i] = 1(p) + 0(1-p)$. Thus,

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = np$$

Expectation of Sums of Random Variables

- **範例五**

Moments of negative binomial random variables

如果獨立試驗成功的機率為p，總共需要達到r次成功，試問期望值為何?

Solution:

If X denotes the number of trials need to amass a total of r successes, then X is a negative binomial random variable that can be represented by

$X = X_1 + X_2 + \cdots + X_r$, where X_1 is the number of trials required to obtain the first success, X_2 is the number of additional trials until the second success is obtained, ….

Expectation of Sums of Random Variables

- That is, X_i represents the number of additional trials required after the $(i-1)$ st success until a total of i successes is amassed.
- A little thought reveals that each of the random variables X_i is a geometric random variable with parameter p. Hence, we know the $E[X_i] = \frac{1}{p}, i = 1,2, \ldots, r$; thus,

$$E[X] = E[X_1] + \cdots + E[X_r] = \frac{r}{p}$$

Expectation of Sums of Random Variables

- **範例六**

Moments of hypergeometric random variables

如果今天從一個摸彩箱(一共含有N顆球且其中有獎的球為m顆)中取n顆球出來，試問: 中獎的期望值為何?

Solution:

Let X denotes the number of prized balls selected, and represent X as

$$X = X_1 + \cdots + X_m, where\ X_i = \begin{cases} 1 & if\ the\ ith\ prized\ ball\ is\ selected \\ 0 & otherwise \end{cases}$$

Expectation of Sums of Random Variables

Now

$$E[X_i] = P\{X_i = 1\} = P\{ith\ prized\ ball\ is\ selected\} = \frac{\binom{1}{1}\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}$$

Hence,

$$E[X] = E[X_1] + \cdots + E[X_m] = \frac{mn}{N}$$

We could have obtained the preceding result by using the alternative representation

$$X = Y_1 + \cdots + Y_n, where\ Y_i = \begin{cases} 1 & if\ the\ ith\ ball\ selected\ is\ prized \\ 0 & otherwise \end{cases}$$

Expectation of Sums of Random Variables

Since the ith ball selected is equally likely to be any of the N balls, it follows that

$$E[Y_i] = \frac{m}{N}$$

So

$$E[X] = E[Y_1] + \cdots + E[Y_n] = \frac{nm}{N}$$

Expectation of Sums of Random Variables

- **範例七**

假設今天畢業典禮有N個人在禮堂拋自己的學士帽，帽子全部混在一起，於是每個人隨機拿一頂，試問: 拿到自己學士帽的期望值。

Solution:

Letting X denote the number of matches, we can compute $E[X]$ most easily by writing

$$X = X_1 + X_2 + \cdots + X_n, where\ X_i = \begin{cases} 1 & if\ the\ ith\ person\ selects\ his\ own\ hat \\ 0 & otherwise \end{cases}$$

Expectation of Sums of Random Variables

Since, for each i, the ith person is equally likely to select any of the N hats,

$$E[X_i] = P\{X_i = 1\} = \frac{1}{N}$$

Thus,

$$E[X] = E[X_1] + \cdots + E[X_N] = \left(\frac{1}{N}\right)N = 1$$

Hence, on the average, exactly one person selects his own hat.

Expectation of Sums of Random Variables

- **範例八**

對於任何一個非負且整數隨機變數X，如果每個$i \geq 1$，則我們可以定義:

$$X_i = \begin{cases} 1 & if\ X \geq i \\ 0 & if\ X < i \end{cases}$$

Then

$$\sum_{i=1}^{\infty} X_i = \sum_{i=1}^{X} X_i + \sum_{i=X+1}^{\infty} X_i = \sum_{i=1}^{X} 1 + \sum_{i=X+1}^{\infty} 0 = X$$

Expectation of Sums of Random Variables

Hence, since the X_i are all nonnegative, we obtain

$$E[X] = \sum_{i=1}^{\infty} E[X_i] = \sum_{i=1}^{\infty} P\{X \geq i\}$$

, which is a useful identity.

Moments of the Number of Events that Occur

- For given events $A_1, \ldots, A_n$, find $E[X]$, where X is the number of these events that occur. The solution then involved defining an indicator variable I_i for event A_i such that

$$I_i = \begin{cases} 1 & if\ A_i\ occurs \\ 0 & otherwise \end{cases}$$

$$\because X = \sum_{i=1}^{n} I_i$$

$$\therefore E[X] = E\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} P(A_i)$$

Moments of the Number of Events that Occur

- Now suppose we are interested in *<u>the number of pairs of events</u>* that occur. Because $I_i I_j$ will equal to 1 if both A_i and A_j occur, and will equal to 0 for otherwise, it follows that the number of pairs is equal to $\sum_{i<j} I_i I_j$. But X is the number of events that occur, it also follows that the number of pairs of events that occur is $\binom{X}{2}$.

$$\binom{X}{2} = \sum_{i<j} I_i I_j$$

Moments of the Number of Events that Occur

Where there are $\binom{n}{2}$ terms in the summation. Taking expectations yields

$$E[(x)] = \sum_{i<j} E[I_i I_j] = \sum_{i<j} P(A_i A_j)$$

Or

$$E\left[\frac{X(X-1)}{2}\right] = \sum_{i<j} P(A_i A_j)$$

Giving that

$$E[X^2] - E[X] = 2\sum_{i<j} P\left(A_i A_j\right) \Rightarrow E[X^2] = 2\sum_{i<j} P\left(A_i A_j\right) + E[X]$$

Moments of the Number of Events that Occur

By considering the number of distinct subsets of k events that all occur, we see that

$$\binom{X}{k} = \sum_{i_1 < i_2 < \cdots < i_k} I_{i_1} I_{i_2} \ldots I_{i_k}$$

Taking expectations gives the identity

$$E\left[\binom{X}{k}\right] = \sum_{i_1 < i_2 < \cdots < i_k} E[I_{i_1} I_{i_2} \ldots I_{i_k}] = \sum_{i_1 < i_2 < \cdots < i_k} P(A_{i_1} A_{i_2} \ldots A_{i_k})$$

Moments of the Number of Events that Occur

- **範例九**

Moments of binomial random variables

假設進行n次獨立試驗，對於每一次試驗成功的機率為p。令A_i為第i次試驗成功的事件。當$i \neq j$，$P(A_i A_j) = p^2$。試問: 其k階動差(moment)為何?

Solution:

$$E\left[\binom{X}{2}\right] = \sum_{i<j} p^2 = \binom{n}{2} p^2, or\ E[X(X-1)] = n(n-1)p^2, or$$

$$E[X^2] - E[X] = n(n-1)p^2$$

$$\Rightarrow E[X^2] = n(n-1)p^2 + E[X] \Rightarrow E[X^2] = n(n-1)p^2 + np$$

Moments of the Number of Events that Occur

Now

$$E[X] = \sum_{i=1}^{n} P(A_i) = np$$

From the preceding equation,

$$Var(X) = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

which is in agreement with the result obtained in the previous.

In general, because $P(A_{i_1} A_{i_2} \cdots A_{i_k}) = p^k$, we can obtain

$$E\left[\binom{X}{k}\right] = \sum_{i_1 < i_2 < \cdots < i_k} p^k = \binom{n}{k} p^k$$

$$\Rightarrow E[X(X-1)\cdots(X-k+1)] = n(n-1)\cdots(n-k+1)p^k$$

Moments of the Number of Events that Occur

[加分題]

已知$E[X(X-1)\cdots(X-k+1)] = n(n-1)\cdots(n-k+1)p^k$

試求$E[X^3]$為何?

Moments of the Number of Events that Occur

- **範例十**

Moments of hypergeometric random variables

假設我們從一個裝有N顆球的摸彩箱中要取出n顆球，其中有m顆是金色(代表中獎)。令A_i為第i顆球為金色的事件。則X為抽到金色球的數量，也可以解釋為在$A_1, A_2, \dots, A_n$所發生的次數。因為每一顆球抽到的機率都相等，所以$P(A_i) = \frac{m}{N}$，期望值為$E[X] = \sum_{i=1}^{n} P(A_i) = \frac{nm}{N}$。試問$X$的$k$階動差為何?

Solution:

$$P(A_i A_j) = P(A_i) P(A_j | A_i) = \frac{m}{N} \frac{m-1}{N-1}$$

Moments of the Number of Events that Occur

$$E\left[\binom{X}{2}\right] = \sum_{i<j} \frac{m}{N}\frac{m-1}{N-1} = \binom{n}{2}\frac{m}{N}\frac{m-1}{N-1}$$

$$E[X(X-1)] = n(n-1)\frac{m}{N}\frac{m-1}{N-1}$$

$$E[X^2] = n(n-1)\frac{m}{N}\frac{m-1}{N-1} + E[X], where\ E[X] = \frac{nm}{N}$$

The formula yields the variance of the hypergeometric,

$$Var(X) = E[X^2] - (E[X])^2 = n(n-1)\frac{m}{N}\frac{m-1}{N-1} + \frac{nm}{N} - \left(\frac{nm}{N}\right)^2$$

$$= \frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1} + 1 - \frac{nm}{N}\right]$$

Moments of the Number of Events that Occur

Therefore, the higher moments of X are obtained …

$$P\left(A_{i_1}A_{i_2}\cdots A_{i_k}\right) = \frac{m(m-1)\cdots(m-k+1)}{N(N-1)\cdots(N-k+1)}$$

$$E\left[\binom{X}{k}\right] = \binom{n}{k}\frac{m(m-1)\cdots(m-k+1)}{N(N-1)\cdots(N-k+1)}$$

Or,

$$E[X(X-1)\cdots(X-k+1)] = n(n-1)\cdots(n-k+1)\frac{m(m-1)\cdots(m-k+1)}{N(N-1)\cdots(N-k+1)}$$

Moments of the Number of Events that Occur

- **範例十一**

假設今天畢業典禮有N個人在禮堂拋自己的學士帽，帽子全部混在一起，每個人隨機拿一頂。令A_i為第i個人拿到自己學士帽的事件，試問其k階動差為何?

Solution:

$$P(A_iA_j) = P(A_i)P(A_j|A_i) = \frac{1}{N}\frac{1}{N-1}$$

$$E\left[\binom{X}{2}\right] = \sum_{i<j}\frac{1}{N(N-1)} = \binom{N}{2}\frac{1}{N(N-1)} \Rightarrow E[X(X-1)] = 1$$

Therefore,

$$E[X^2] = 1 + E[X], since\ E[X] = \sum_{i=1}^{N}P(A_i) = 1 \Rightarrow E[X^2] = 2$$

Moments of the Number of Events that Occur

We obtain that

$$Var(X) = E[X^2] - (E[X])^2 = 1$$

Hence, both expected value (mean) and variance of the number of matches is 1.

For higher moment , we can obtain…

$$P\left(A_{i_1} A_{i_2} \cdots A_{i_k}\right) = \frac{1}{N(N-1)\cdots(N-k+1)}$$

$$E\left[\binom{X}{k}\right] = \binom{N}{k} \frac{1}{N(N-1)\cdots(N-k+1)}$$

$$E[X(X-1)\cdots(X-k+1)] = 1$$

Covariance, Variance of Sums, and Correlations

- The expectation of a product of independent random variables is equal to the product of their expectations.
- **Proposition 2**

If X and Y are independent, then, for any functions h and g.

$$E[g(X)h(Y)] = E[g(X)]E[h(X)]$$

Proof:

$$E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x)f(y)dxdy = \int_{-\infty}^{\infty} g(x)f(x)dx \int_{-\infty}^{\infty} h(y)f(y)dy$$

$$= E[g(X)]E[h(Y)]$$

Covariance, Variance of Sums, and Correlations

- **Definition**

The covariance between X and Y, dented by $Cov(X,Y)$, is defined by

$$Cov(X,Y) = E[(X - E[X])]\,E[(Y - E[Y])]$$

Upon expanding the right side of the preceding definition,

$$Cov(X,Y) = E\big[XY - E[X]Y - XE[Y] + E[Y]E[X]\big]$$
$$Cov(X,Y) = E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y]$$
$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

If X and Y are independent, then, by **Proposition 2**, $Cov(X,Y) = 0$.

Covariance, Variance of Sums, and Correlations

- However, the converse is not true. A simple case of two dependent random variables X and Y having zero covariance is obtained by letting X be a random variable such that

$$P\{X=0\}=P\{X=1\}=P\{X=-1\}=\frac{1}{3}$$

- By defining

$$Y=\begin{cases}0 & if\ X\neq 0\\ 1 & if\ X=0\end{cases}$$

- Now, $XY=0$, so $E[XY]=0$. Also, $E[X]=0$. Thus,

$$Cov(X,Y)=E[XY]-E[X]E[Y]=0$$

Covariance, Variance of Sums, and Correlations

- **Proposition 3**

- $Cov(X, Y) = Cov(Y, X)$
- $Cov(X, X) = Var(X)$
- $Cov(aX, Y) = aCov(X, Y)$
- $Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$

Covariance, Variance of Sums, and Correlations

Proof:

(1) $Cov(X, Y) = Cov(Y, X)$

(2) $Cov(X, X) = Var(X)$

(3) $Cov(aX, Y) = aCov(X, Y)$

These three could be derived from the definition of covariance.

Let $\mu_i = E[X_i]$ and $v_j = E[Y_j]$

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mu_i\,, E\left[\sum_{j=1}^{m} Y_j\right] = \sum_{j=1}^{m} v_j$$

Covariance, Variance of Sums, and Correlations

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} v_j\right)\right]$$

$$= E\left[\sum_{i=1}^{n}(X_i - \mu_i)\sum_{j=1}^{m}(Y_j - v_j)\right] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(X_i - \mu_i)(Y_j - v_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} E\left[(X_i - \mu_i)(Y_j - v_j)\right]$$

Covariance, Variance of Sums, and Correlations

$$Var\left(\sum_{i=1}^{n} X_i\right) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} Var(X_i) + \sum\sum_{i \neq j} Cov(X_i, X_j)$$

Since each pair of indices i, j, $i \neq$ j, appears twice in the double summation, the preceding formula is equivalent to

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i, X_j)$$

Covariance, Variance of Sums, and Correlations

If $X_1, \ldots, X_n$ are pairwise independent, in that X_i and X_j are independent for $i \neq j$, then we can reduce the formula from

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i, X_j)$$

to

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

Since

$$2\sum\sum_{i<j} Cov(X_i, X_j) = 0$$

Covariance, Variance of Sums, and Correlations

- 範例十二

令 $X_1, \ldots, X_n$ 為 independent and identical distributed random variables，其參數μ與σ^2。樣本平均數為$\bar{X} = \sum_{i=1}^{n} X_i/n$。現在定義一個deviations為$X_i - \bar{X}, i = 1, \ldots, n$。我們可以定義新的隨機變數為

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}$$

這個稱做<u>樣本變異數</u>(sample variance)。試問: $Var(\bar{X})$ 與 $E[S^2]$。

Covariance, Variance of Sums, and Correlations

- $Var(\bar{X})$

$$Var(\bar{X}) = \left(\frac{1}{n}\right) Var\left(\sum_{i=1}^{n} X_i\right) = \left(\frac{1}{n}\right) \sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$$

- The following algebraic identity

$$(n-1)S^2 = \sum_{i=1}^{n} (X_i - \mu + \mu - \bar{X})^2$$

$$= \sum_{i=1}^{n} (X_i - \mu)^2 + \sum_{i=1}^{n} (\bar{X} - \mu)^2 - 2(\bar{X} - \mu) \sum_{i=1}^{n} (X_i - \mu)$$

Covariance, Variance of Sums, and Correlations

$$= \sum_{i=1}^{n} (X_i - \mu)^2 + \sum_{i=1}^{n} (\bar{X} - \mu)^2 - 2(\bar{X} - \mu) \sum_{i=1}^{n} (X_i - \mu)$$

$$= \sum_{i=1}^{n} (X_i - \mu)^2 + \sum_{i=1}^{n} (\bar{X} - \mu)^2 - 2(\bar{X} - \mu) n(\bar{X} - \mu)$$

$$= \sum_{i=1}^{n} (X_i - \mu)^2 - n(\bar{X} - \mu)^2$$

Take expectations of the preceding yields

$$(n-1)E[S^2] = \sum_{i=1}^{n} E[(X_i - \mu)^2] - nE[(\bar{X} - \mu)^2] = n\sigma^2 - nVar(\bar{X}) = (n-1)\sigma^2$$

Covariance, Variance of Sums, and Correlations

- 範例十三

計算binomial random variable $X\ (n,p)$的變異數。

Solution:

Since a random variable represents the number of successes in n independent trials when each trial has the common probability p of being a success, we may write

$$X = X_1 + \cdots + X_n$$

Where the X_i are independent Bernoulli random variables such that

$$X_i = \begin{cases} 1 & \text{if the ith trial is a success} \\ 0 & \text{otherwise} \end{cases}$$

Covariance, Variance of Sums, and Correlations

Since we know …

$$Var(X) = Var(X_1) + \cdots + Var(X_n)$$

But

$$Var(X_i) = E[X_i^2] - (E[X])^2$$

Since $X_i^2 = X_i$, therefore,

$$Var(X_i) = E[X_i] - (E[X])^2$$
$$= p - p^2$$

Thus

$$Var(X) = np(1 - p)$$

Covariance, Variance of Sums, and Correlations

- **範例十四**

令I_A與I_B為事件A與B的指標變數，故

$$I_A = \begin{cases} 1 & if\ A\ occurs \\ 0 & otherwise \end{cases}$$

$$I_B = \begin{cases} 1 & if\ B\ occurs \\ 0 & otherwise \end{cases}$$

$$E[I_A] = P(A), E[I_B] = P(B), E[I_A I_B] = P(AB)$$

$$Cov(I_A, I_B) = P(AB) - P(A)P(B)$$
$$= P(B)[P(A|B) - P(A)]$$

Covariance, Variance of Sums, and Correlations

- 範例十五

令 $X_1, \ldots, X_n$ 為 independent and identically distributed random variables having variance σ^2. Show that

$$Cov(X_i - \bar{X}, \bar{X}) = 0$$

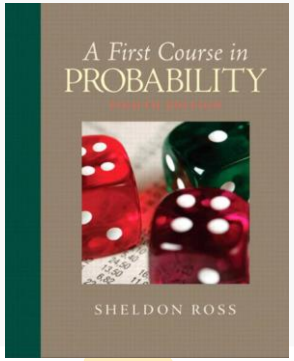
Solution:

$$Cov(X_i - \bar{X}, \bar{X}) = Cov(X_i, \bar{X}) - Cov(\bar{X}, \bar{X})$$

$$= Cov\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) - Var(\bar{X}) = \frac{1}{n}\sum_{j=1}^{n} Cov\left(X_i, X_j\right) - \frac{\sigma^2}{n} = \frac{\sigma^2}{n} - \frac{\sigma^2}{n}$$

$$= 0$$

[#13] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

7.1. A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then she wins twice, and if tails, then one-half of the value that appears on the die. Determine her expected winnings.

7.4. If X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 1/y, & \text{if } 0 < y < 1,\ 0 < x < y \\ 0, & \text{otherwise} \end{cases}$$

find

(a) $E[XY]$
(b) $E[X]$
(c) $E[Y]$

7.6. A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.

7.7. Suppose that A and B each randomly and independently choose 3 of 10 objects. Find the expected number of objects

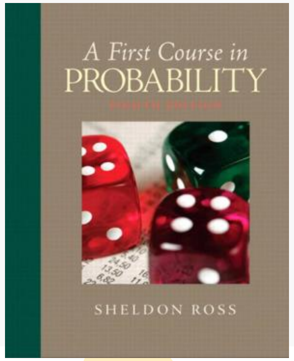
(a) chosen by both A and B;
(b) not chosen by either A or B;
(c) chosen by exactly one of A and B.

7.9. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find

(a) the expected number of urns that are empty;
(b) the probability that none of the urns is empty.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

[#13] Assignment



7.30. If X and Y are independent and identically distributed with mean μ and variance σ^2, find

$$E[(X - Y)^2]$$

7.33. If $E[X] = 1$ and $\text{Var}(X) = 5$, find
(a) $E[(2 + X)^2]$;
(b) $\text{Var}(4 + 3X)$.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

Reference

Ross, S. (2010). *A first course in probability*. Pearson.

The End

If you have any questions, please do not hesitate to ask me.

Thank you for your attention))